The application of the Kirchhoff approximation to the initial mathematical model allows us to solve the problem on a regular grid, since the boundary conditions (6) are transformed into the equality of the derivatives of the new functions with respect to the normal. In this case at the point of contact there is a step between the new functions, which increases as the ratio of the thermal conductivities of the materials in contact increases. The IHC problem is best solved (from the standpoint of computational difficulties) by using the method when the observation points are arranged in one of th ebodies in contact.

In summary, the proposed method of solving the IHC problem on the basis of limited information about the thermal state of the composite body allows the TRC to be calculated with engineering accuracy by numerical methods, the use of which is necessitated, as a rule, by the complex geometry of the objects studied. The most promising for solving problems of this class is the regional-structural method [2], which allows the heat flux to be determined in a continuous form.

NOTATION

Here T denotes the temperature; x and y are the spatial coordinates; t is the time coordinate; q is the heat flux, a_{ij} and b_{ij} are the parameters of the boundary-effect functions; W_{ij} are the spectral boundary-effect fucntions; c_v is the specific heat at constant volume; λ is the thermal conductivity; α is the heat-exchange coefficient; and R is the thermal contact resistance. Indices: s is the number of the observation point; sur denotes surface; and m denotes medium.

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APPLICATION OF DIRICHLET AND NEUMANN PROBLEMS IN CONNECTION WITH STUDIES OF NONSTATIONARY HEAT CONDUCTION

V. P. Kozlov, V. S. Adamchuk, and V. N. Lipovtsev

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Regularities in the development of three-dimensional nonstationary temperature fields in semi-bounded iso- and orthotropic media are deduced under discontinuous boundary conditions of the first or second kind, given in the most general form.

A theoretical foundation for the creation of modern methods and measuring tools for the nondestructive control of thermophysical characteristics (TPC) of various materials is furnished by appropriate solutions of many-dimensional nonstationary problems of heat conduction with discontinuous boundary conditions (BC) [1-22]. As a result of the action of arbitrary discontinuous BC on surfaces of a medium being investigated, temperature fields arising directly from a boundary surface (in a region of action of discontinuous BC) will carry thermophysical information concerning the whole complex of TPC for the given medium. This latter circumstance makes it possible to organize complex thermophysical measurements of various materials without invading its intrinsic structure (the so-called methods and means of nondestructive control of TPC).

In the present paper we consdier the classical formulations of Dirichlet and Neumann problems in a nonstationary version and apply them to the solution of corresponding axially-

Gosplan BSSR Scientific-Research Economics Institute, Minsk.

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symmetric problems of nonstationary heat conduction for semi-bounded iso- and orthotropic media with discontinuous BC of the general form:

$$\Theta(r,\ z,\ \tau)|_{z=0}=\begin{cases}\Theta_{1\text{п}}(r,\ \tau) & \text{for}\quad r<R_1,\\ \Theta_{2\text{п}}(r,\ \tau) & \text{for}\quad R_1<r<R_2,\\ 0 & \text{for}\quad R_2<r<\infty\end{cases} \tag{1}$$

or

$$-\frac{\partial\Theta(r,\ z,\ \tau)}{\partial z}\bigg|_{z=0}=\begin{cases}\dfrac{q_{1\text{п}}(r,\ \tau)}{\lambda_z} & \text{for}\quad r<R_1,\\ \dfrac{q_{2\text{п}}(r,\ \tau)}{\lambda_z} & \text{for}\quad R_1<r<R_2,\\ 0 & \text{for}\quad R_2<r<\infty,\end{cases} \tag{2}$$

where $\Theta_{1\pi}(r,\ \tau)$, $\Theta_{2\pi}(r,\ \tau)$, $q_{1\pi}(r,\ \tau)$, $q_{2\pi}(r,\ \tau)$ are given functions of surplus temperature and density of thermal flows on the boundary ($z = 0$) of a semi-bounded (orthotropic) body in corresponding ranges of variation in the cylindrical coordinate r. No special restrictions are imposed on the discontinuous functions $\Theta_{i\pi}(r,\ \tau)$ and $q_{i\pi}(r,\ \tau)$ ($i = 1,\ 2$) in the boundary conditions (1) and (2) except for the existence and convergence of the following integrals:

$$\int_0^\infty\int_0^{R_1} rJ_0(rt)\exp(-s\tau)\begin{Bmatrix}\Theta_{1\text{п}}(r,\ \tau)\\ q_{1\text{п}}(r,\ \tau)\end{Bmatrix}drd\tau;\quad \int_0^\infty\int_{R_1}^{R_2} rJ_0(rt)\exp(-s\tau)\begin{Bmatrix}\Theta_{2\text{п}}(r,\ \tau)\\ q_{2\text{п}}(r,\ \tau)\end{Bmatrix}drd\tau.$$

In what follows we shall be concerned with solving the following heat conduction equation:

$$\frac{\partial^2\Theta(r,\ z,\ \tau)}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta(r,z,\ \tau)}{\partial r}+\frac{a_z}{a_r}\frac{\partial^2\Theta(r,\ z,\ \tau)}{\partial z^2}=\frac{1}{a_r}\frac{\partial\Theta(r,\ z,\ \tau)}{\partial\tau}. \tag{3}$$

In L-transform space the solution of Eq. (3), subject to BC specified in the form (1), can be written in the form

$$\bar\Theta(r,\ z,\ s)=\int_0^\infty tJ_0(rt)\left\{\int_0^{R_1} x\bar\Theta_{1\text{п}}(x,\ s)J_0(xt)\,dx+\int_{R_1}^{R_2} x\bar\Theta_{2\text{п}}(x,\ s)J_0(xt)\,dx\right\}\exp\left(-\frac{z\sqrt{s+a_rt^2}}{\sqrt{a_z}}\right)dt, \tag{4}$$

$$\bar\Theta_{i\text{п}}(x,\ s)=\int_0^\infty\Theta_{i\text{п}}(x,\ \tau)\exp(-s\tau)\,d\tau,\quad i=1,\ 2.$$

In the time domain the solution (4) may be written in the form

$$\Theta(r,\ z,\ \tau)=\frac{1}{2}\frac{d}{d\tau}\int_0^\tau\int_0^\infty tJ_0(rt)\left\{\int_0^{R_1} x\Theta_{1\text{п}}(x,\ \tau-\xi)J_0(xt)\,dx+\int_{R_1}^{R_2} x\Theta_{2\text{п}}(x,\ \tau-\xi)J_0(xt)\,dx\right\}\left\{\exp(-zt\sqrt{K_a})\times\right.$$
$$\times\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\xi}}-t\sqrt{a_r\xi}\right)+\exp(zt\sqrt{K_a})\times$$
$$\left.\times\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\xi}}+t\sqrt{a_r\xi}\right)\right\}dtd\xi, \tag{5}$$

where $K_a = a_r/a_z = \lambda_r/\lambda_z = K_\lambda$.

We consider a particular instance of the solution (5) for the case of heat exchange for the body in question for constant, but not identical, temperatures $\Theta_{1\pi}(r,\ \tau) + T_c - T_0 =$ const and $\Theta_{2\pi}(r,\ \tau) = T_c^* - T_0 = \text{const}(T_c \neq T_c^* \neq T_0)$ in specified regions on its bounding surface:

$$\Theta(r,\ z,\ \tau)=\frac{1}{2}\int_0^\infty\left\{(T_c^*-T_0)J_1(x)+(T_c-T_c^*)\frac{R_1}{R_2}J_1\left(\frac{R_1}{R_2}x\right)\right\}\times$$

$$\times\left\{\exp\left(-\frac{z\sqrt{K_a}}{R_2}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}-\frac{\sqrt{a_r\tau}}{R_2}x\right)+\right.$$

$$\left.+\exp\left(\frac{z\sqrt{K_a}}{R_2}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}+\frac{\sqrt{a_r\tau}}{R_2}x\right)\right\}J_0\left(\frac{r}{R_2}x\right)dx. \tag{6}$$

The well-known one-dimensional solution for BC of the first kind [21] is readily obtained from Eq. (6) for $R_1 = R_2 = R \to \infty$ or for $T_c^* = T_c \neq T_0$ and $R_2 \to \infty$.

Let us assume that in solution (6) we have $T_c^* = T_0$, which corresponds to maintaining in the region $0 \le r < R_1$ $(z = 0)$ a constant temperature $T_c \neq T_0$, and in the region $\infty > r > R_1$ $(z = 0)$ an initial temperature T_0. Then

$$\Theta(r,\ z,\ \tau)=\frac{T_c-T_0}{2}\int_0^\infty\left\{\exp\left(-\frac{z\sqrt{K_a}}{R_1}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}-\frac{\sqrt{a_r\tau}}{R_1}x\right)+\right.$$

$$\left.+\exp\left(\frac{z\sqrt{K_a}}{R_1}x\right)\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}+\frac{\sqrt{a_r\tau}}{R_1}x\right)\right\}J_1(x)J_0\left(\frac{r}{R_1}x\right)dx. \tag{7}$$

The simplest solution, starting from Eq. (7), is that on the axis $r = 0$:

$$\Theta(0,\ z,\ \tau)=T(0,\ z,\ \tau)-T_0=(T_c-T_0)\left\{\operatorname{erfc}\left(\frac{z}{2\sqrt{a_z\tau}}\right)-\right.$$

$$\left.-\frac{1}{\sqrt{1+\frac{R_1^2}{z^2K_a}}}\operatorname{erfc}\left(\frac{R_1}{2\sqrt{a_r\tau}}\sqrt{1+\frac{z^2K_a}{R_1^2}}\right)\right\}. \tag{8}$$

Using Eq. (8), we find the specific thermal flow $q(0,\ \tau) = q(0,\ 0,\ \tau) = \lambda_z\frac{\partial\Theta(0,\ z,\ \tau)}{\partial z}$ on the boundary $z = 0$, originating at the central point $(r = z = 0)$ of the circular isothermal probe:

$$q(0,\ \tau)|_{r=z=0}=\frac{b_z(T_c-T_0)}{\sqrt{\pi\tau}}\left\{1+\frac{\sqrt{\pi a_r\tau}}{R_1}\operatorname{erfc}\left(\frac{R_1}{2\sqrt{a_r\tau}}\right)\right\}, \tag{9}$$

where $b_z = \lambda_z/\sqrt{a_z}$ is the thermal activity in the direction of the z-axis.

As $R_1 \to \infty$, equation (9) yields the relation for the one-dimensional (per unit area) thermal flow $q(\tau)$ originating from the surface $(z = 0)$ of an iso- and orthotropic halfspace at the expense of maintaining, on the boundary of the body considered, a constant temperature T_c different from the initial temperature T_0 [21].

The two-dimensional expression (9) can be used to calculate parameters of TPC by the circular isothermal probe method [1, 23]. We note that as $\tau \to \infty$ (stationary regime), expression (9) yields the original formula for calculation of TPC complexes:

$$\lambda_z\sqrt{K_\lambda}=\frac{\lambda_r}{\sqrt{K_\lambda}}=\sqrt{\lambda_z\lambda_r}=b_z\sqrt{a_r}=b_r\sqrt{a_z}=$$

$$=\lambda_z\sqrt{K_a}=\frac{\lambda_r}{\sqrt{K_a}}=c\gamma\sqrt{a_za_r}=\frac{q_{cT}R_1}{\Delta T_{cT}}, \tag{10}$$

where $q_{cT} = q(0,\ 0,\ \infty)$ is the stationary value of the thermal flow density at the central point $(r = z = 0)$ of the isothermal disk; $\Delta T_{cT} = T_c - T_0$; $c\gamma$ is the volumetric thermal capacity of the orthotropic body.

The classical solution of Eq. (3) in L-transform space, subject to the boundary conditions (2), can be written in the following form:

$$\bar{\Theta}(r,\ z,\ s)=\frac{1}{b_z}\int_0^\infty tJ_0(rt)\frac{\exp\left[-\frac{z\sqrt{s+a_rt^2}}{\sqrt{a_z}}\right]}{\sqrt{s+a_rt^2}}\left\{\int_0^{R_1}x\bar{q}_{1\text{п}}(x,\ s)J_0(xt)dx+\int_{R_1}^{R_2}x\bar{q}_{2\text{п}}(x,\ s)J_0(xt)dx\right\}dt, \tag{11}$$

where

$$\bar{q}_{i\text{п}}(x,\ s) = \int_0^\infty q_{i\text{п}}(x,\ \tau)\exp(-s\tau)\,d\tau;\ i = 1,\ 2.$$

In the time domain solution (11) may be written in the form:

$$\Theta(r,\ z,\ \tau) = \frac{1}{2\lambda_z \sqrt{K_a}}\ \frac{d}{d\tau}\int_0^\tau\int_0^\infty J_0(rt)\left\{\int_0^{R_1} xJ_0(xt)\,q_{1\text{п}}(x,\ \tau-\xi)\,dx + \right.$$

$$\left. + \int_{R_1}^{R_2} xJ_0(xt)\,q_{2\text{п}}(x,\ \tau-\xi)\,dx\right\}\left\{\exp(-zt\sqrt{K_a})\,\mathrm{erfc}\left(\frac{z}{2\sqrt{a_z\xi}} - t\sqrt{a_r\xi}\right) - \right.$$

$$\left. - \exp(zt\sqrt{K_a})\,\mathrm{erfc}\left(\frac{z}{2\sqrt{a_z\xi}} + t\sqrt{a_r\xi}\right)\right\}dt\,d\xi. \qquad (12)$$

Specifying in these regions on the surface (z = 0) of an orthotropic halfspace a concrete law of variation for the density of the thermal flows $q_{1\text{п}}(r,\ \tau)$ and $q_{2\text{п}}(r,\ \tau)$, we obtain, by a simple integration of the right sides of equations (11) or (12), a series of particular solutions for determining the corresponding temperature fields $\Theta(r,\ z,\ s)$ or $\Theta(r,\ z,\ \tau)$ [1-17, 19, 20].

In determining the corresponding analytic functions (4), (5), (11), (12), with concretely specified discontinuous functions of temperature or thermal flow density on the boundary of the body considered, the following identical integral transformations prove to be useful:

$$\int_0^\infty \cos pz K_0\left(R\sqrt{p^2+\frac{s}{a}}\right)I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)dp =$$

$$= \frac{\pi}{2\sqrt{a}}\int_0^\infty \frac{y}{\sqrt{s+y^2}}\exp\left(-\frac{z\sqrt{s+y^2}}{\sqrt{a}}\right)J_0\left(\frac{ry}{\sqrt{a}}\right)J_0\left(\frac{Ry}{\sqrt{a}}\right)dy,\ R \geqslant r;$$

$$\int_0^\infty p\sin pz K_0\left(R\sqrt{p^2+\frac{s}{a}}\right)I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)dp =$$

$$= \frac{\pi}{2a}\int_0^\infty y\exp\left(-\frac{z\sqrt{s+y^2}}{\sqrt{a}}\right)J_0\left(\frac{ry}{\sqrt{a}}\right)J_0\left(\frac{Ry}{\sqrt{a}}\right)dy,\ R > r;$$

$$\int_0^\infty \frac{\cos pz}{\sqrt{p^2+s/a}}K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)I_1\left(R\sqrt{p^2+\frac{s}{a}}\right)dp =$$

$$= \frac{\pi}{2}\int_0^\infty \exp\left(-\frac{z\sqrt{s+y^2}}{\sqrt{a}}\right)\frac{J_1\left(\frac{Ry}{\sqrt{a}}\right)J_0\left(\frac{ry}{\sqrt{a}}\right)}{\sqrt{s+y^2}}dy,\ r \geqslant R;$$

$$\int_0^\infty \frac{\cos pz}{\sqrt{p^2+s/a}}K_1\left(R\sqrt{p^2+\frac{s}{a}}\right)I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)dp =$$

$$= \frac{\pi\sqrt{a}}{2R\sqrt{s}}\exp\left(-\frac{z\sqrt{s}}{\sqrt{a}}\right) - \frac{\pi}{2}\int_0^\infty \frac{\exp\left(-\frac{z\sqrt{s+y^2}}{\sqrt{a}}\right)}{\sqrt{s+y^2}}J_1\left(\frac{Ry}{\sqrt{a}}\right)\times$$

$$\times J_0\left(\frac{ry}{\sqrt{a}}\right)dy,\ R \geqslant r.$$

NOTATION

T_0 = const, initial temperature of the orthotropic halfspace of the body considered; $\theta(r,\ z,\ \tau) = T(r,\ z,\ \tau) - T_0$, surplus temperature; $T(r,\ z,\ \tau)$, value of nonstationary temperature field of orthotropic mass; r, z, cylindrical coordinates; τ, time; $\theta_{i\text{п}}(r,\ \tau) = T_{i\text{п}}(r,\ \tau) - T_0$ and $q_{i\text{п}}(r,\ \tau)$, respective discontinuous functions of surplus temperature and thermal flow density, given on boundary z = 0 of orthotropic halfspace in corresponding regions of variation of cylindrical coordinate r, (i = 1, 2); $J_0(x)$, $J_1(x)$, Bessel functions of a real argument of orders zero an done; $I_0(x)$, $K_0(x)$, $I_1(x)$, $K_1(x)$, modified Bessel functions of orders zero and one; a_r, a_z, λ_r, λ_z, b_r, b_z, respective diffusivity, thermal conductivity, and thermal activity of orthotropic body in direction of r and z axes; $K_a = a_r/a_z$; $K_\lambda = \lambda_r/\lambda_z$; $K_b = b_r/b_z$, dimensionless parameters characterizing relationships among thermophysical properties of orthotropic body; $c_z\gamma_z = c_r\gamma_r = c\gamma$, constant temperatures specified on surface z = 0 of orthotropic mass in corresponding ranges of variation for r; erfc x = 1 − erf x, complementary probability integral; $q(0, \tau) = q(0, 0, \tau)$, thermal flow density at center (r = z = 0) of isothermal disk placed on surface z = 0 of orthotropic mass (two-dimensional case).

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